Example of complex integral.

Find $\int_{\alpha} z^2 dz$ where $\alpha(t) = e^{it}$ for $0 \le t \le \frac{\pi}{2}$.

Solution $z = \alpha(t) = e^{it}$

$z^2 = e^{2it}$

$dz = d(e^{it}) = ie^{it}$

$$\int_{t=0}^{\pi/2} \left(e^{2it}\right)\left(ie^{it}\right)dt$$

$$= \int_0^{\pi/2} i\, e^{3it}\, dt = i \left.\frac{e^{3it}}{3i}\right|_0^{\pi/2}$$

$$\left(\frac{e^{3it}}{3i}\right)' = e^{3it}$$

$$= \frac{e^{3i\pi/2}}{3} - \frac{e^0}{3}$$

$$= \boxed{-\frac{i}{3} - \frac{1}{3}}.$$

Real variable method:

$z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy)$

$dz = dx + i\,dy$

$$\int_{\alpha}\left[(x^2-y^2)+i(2xy)\right]\left[dx+i\,dy\right]$$

$$= \int_{\alpha}(x^2-y^2)\,dx - 2xy\,dy$$

$$+ i\int_{\alpha} 2xy\,dx + (x^2-y^2)\,dy$$

$\alpha(t) = (\cos(t), \sin(t))$ for $(0 \le t \le \frac{\pi}{2})$

$\alpha'(t) = (x'(t), y'(t)) = (-\sin(t), \cos(t))$

$$= \int_{t=0}^{\pi/2} \left(\cos^2(t) - \sin^2(t)\right)\left(-\sin(t)\right)dt$$

$$- 2\left(\cos(t)\sin(t)\right)\left(\cos(t)\right)dt$$

$$+ i\left[\int_0^{\pi/2} 2\left(\cos(t)\sin(t)\cos(t)\right)dt\right.$$

$$\left.+ \int_0^{\pi/2}\left(\cos^2(t)-\sin^2(t)\right)\cos(t)\,dt\right]$$

$$= \int_0^{\pi/2}\left[-\cos^2(t)\sin(t) + \sin^3(t) - 2\cos^2(t)\sin(t)\right.$$

$$+ i\left(2\cos^2(t)\sin(t)\right) + i\cos^3(t)$$

$$\left.- i\cos(t)\sin^2(t)\right]dt.$$

$= \ldots\ldots = \cdot$

$$\int_0^{\pi/2} \sin^3(t)\,dt = \int_0^{\pi/2} (1-\cos^2(t))\sin(t)\,dt$$

$u = \cos(t)$

$du = -\sin(t)\,dt$

$$= \int_1 (u^2 - 1)\,du = \frac{u^3}{3} - u \Big|_1^0$$

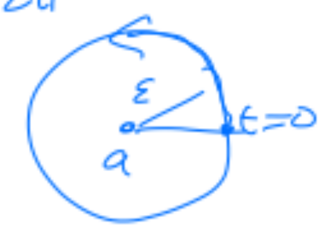
$$= -\frac{1}{3} - 1 = -\frac{4}{3}.$$

Important Example:

Find
$$\int_{\partial B(a,\varepsilon)} (z-a)^n\,dz = I \qquad n \in \mathbb{Z}$$

Circle of radius $\varepsilon > 0$ centered at a oriented CCW

$$z = a + \varepsilon e^{it}, \quad 0 \le t \le 2\pi$$



$$dz = \varepsilon i e^{it}\,dt$$

$$I = \int_0^{2\pi} (\varepsilon e^{it})^n (\varepsilon i e^{it}\,dt)$$

$$= \varepsilon^{n+1} i \int_0^{2\pi} e^{i(n+1)t}\,dt$$

$$= \varepsilon^{n+1} i \int_0^{2\pi} \left(\cos((n+1)t) + i \sin((n+1)t) \right) dt$$

$$= \varepsilon^{n+1} i \begin{cases} \left. \dfrac{\sin((n+1)t)}{n+1} - i \dfrac{\cos((n+1)t)}{n+1} \right|_0^{2\pi} & n \neq -1 \\ \left. t \right|_0^{2\pi} & n = -1 \end{cases}$$

$$= \varepsilon^{n+1} \cdot i \cdot \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi & \text{if } n = -1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$$

$\varepsilon^{(-1)+1} = \varepsilon^0 = 1$

for $n \in \mathbb{Z}$
$$\int_{\partial B(a,\varepsilon)} (z-a)^n \, dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1. \end{cases}$$

Independent of ε.

Fundamental Theorem of Calculus for Line Integral (FTCLI):

If α is a piecewise smooth curve in $\mathbb{R}^n$, and f is a differentiable function of n-variables, then

$$\oint_\alpha df = \oint_\alpha \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$$

$(\alpha : [a,b] \to \mathbb{R}^n)$

$$= f(\underbrace{\alpha(b)}_{\text{last point on curve}}) - f(\underbrace{\alpha(a)}_{\text{initial point of } \alpha}).$$

Proof: $\alpha(t) = (x_1(t), \dots, x_n(t))$

$\alpha'(t) = (x_1'(t), \dots, x_n'(t))$

$dx_j = x_j'(t)\,dt$ etc. plug in and use FTC. $\boxed{}$

$$d\{f(\alpha(t))\} = \underbrace{\nabla f(\alpha(t)) \cdot \alpha'(t)}_{\frac{d}{dt}(f(\alpha(t)))\,dt.}\,dt = f_{x_1} dx_1 + \dots + f_{x_n} dx_n$$

Now apply this to complex integrals.

FTC for Complex Integrals. (FTCCI)

If f is complex differentiable on the domain $D \subseteq \mathbb{C}$, and if α is a piecewise smooth curve in D, and f' is an integrable fcn, then

$$\int_\alpha f'(z)\,dz = f(\alpha(b)) - f(\alpha(a)).$$

$\alpha : [a,b] \to D \subseteq \mathbb{C}$

Proof: $d(f(z)) = f'(z)\,dz$,

[ie. same calculation $f(x+iy) = u(x,y) + iv(x,y)$.
$\rightarrow L = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = f'(z)$.]

Let's do our first integral again:

Find $\int_\alpha z^2 dz$ where $\alpha(t) = e^{it}$ for $0 \le t \le \frac{\pi}{2}$.

New solution: $\alpha(0) = 1$, $\alpha(\frac{\pi}{2}) = i$

$$\int_\alpha z^2 dz = \left(\frac{z^3}{3}\right)\Bigg|_1^i = \frac{(i)^3}{3} - \frac{1}{3}$$

$$= \boxed{\frac{-i}{3} - \frac{1}{3}}.$$

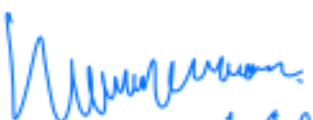
Corollary: If α is a closed loop, then $\int_\alpha f'(z) dz = 0$.

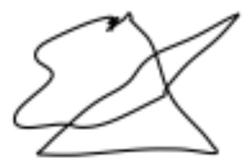
$(\alpha(a) = \alpha(b))$.

Note: If α is a closed, piecewise smooth curve in $\mathbb{C}$, and if z_0 is a point in $\mathbb{C}$, the winding number of α going around z_0

is $\boxed{\frac{1}{2\pi i}\int_{\alpha} \frac{1}{(z-z_0)}\, dz}$ We will find out why this makes sense later.

We say that a piece-wise smooth curve is <u>rectifiable</u> if its length is finite.

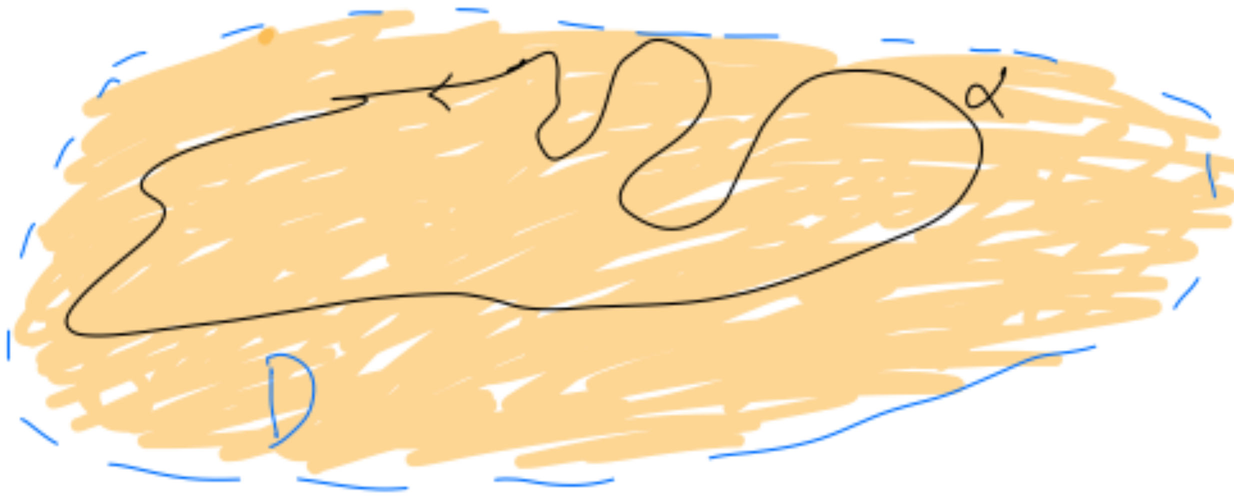

not rectifiable


rectifiable.

<u>Important Theorem</u>

<u>Cauchy Theorem (Cauchy Goursat Thm)</u>

If f is holomorphic on a <u>domain</u> (open connected set.) D, and if α is a continuous, simple, rectifiable, closed curve in D whose interior is inside D, then $\int_{\alpha} f(z)\,dz = 0$.



(Note: this is clear if $f(z) = g'(z)$ for some holomorphic fcn g on D, by the FTCCI.)

Why is this true?

- Easier proof: use Green's Theorem, which works if α is piecewise smooth.

$$\oint_{\alpha \text{ ccw curve}} F(x,y)\,dx + G(x,y)\,dy$$

$$= \int_{\substack{\text{interior of } \alpha \\ \text{2-d}}} d\big(F(x,y)\,dx + G(x,y)\,dy\big)$$

$$\overset{\text{Green's Theorem}}{=} \int (-F_y + G_x)\,dx \wedge dy$$

$$d(\;) = (F_x\,dx + F_y\,dy) \wedge dx + (G_x\,dx + G_y\,dy) \wedge dy$$

$dx \wedge dx = 0 = dy \wedge dy$
$dx \wedge dy = -dy \wedge dx$

$$= -F_y\,dx \wedge dy + G_x\,dx \wedge dy$$

Proof: next time